ON SOME CLASSES OF LINDELÖF Σ-SPACES

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Abstract. We consider special subclasses of the class of Lindelöf Σ-spaces obtained by imposing restrictions on the weight of the elements of compact covers that admit countable networks: A space X is in the class $L\Sigma(\leqslant\kappa)$ if it admits a cover by compact subspaces of weight κ and a countable network for the cover. We restrict our attention to $\kappa\le\omega$. In the case $\kappa=\omega$, the class includes the class of metrizably fibered spaces considered by Tkachuk, and the P-approximable spaces considered by Tkačenko. The case $\kappa=1$ corresponds to the spaces of countable network weight, but even the case $\kappa=2$ gives rise to a nontrivial class of spaces. The relation of known classes of compact spaces to these classes is considered. It is shown that not every Corson compact of weight $\aleph_1$ is in the class $L\Sigma(\le\omega)$, answering a question of Tkachuk. As well, we study whether certain compact spaces in $L\Sigma(\le\omega)$ have dense metrizable subspaces, partially answering a question of Tkačenko. Other interesting results and examples are obtained, and we conclude the paper with a number of open questions.

1. Preliminaries

All spaces we consider are Tychonoff (that is, completely regular Hausdorff), unless otherwise indicated. We use terminology and notation as in [5], with the exception that the tightness of a space X is denoted as $t(X)$.

Given a locally compact space X, we denote by αX its one-point compactification, the new point will be usually denoted by ∞ (unless $\infty\in X$). The one-point compactification of a discrete space of size κ will be denoted by $A(\kappa)$.

A topological space X is a *Lindelöf Σ-space*, if it is a Σ-space in the sense of Nagami [9] and is Lindelöf. Lindelöf Σ-spaces are also known as *K-countably determined spaces* [12].

A *multivalued mapping* from X to Y is a mapping that assigns to every point of X a subset of Y (not necessarily nonempty). For a multivalued mapping $p\colon X\to Y$ and a set $A\subset X$, the *image* of A under p is the set

$$p(A)=\bigcup\{p(x):x\in A\};$$

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a mapping $p : X \to Y$ is *onto* if $p(X) = Y$. If $p\colon X \to Y$ and $q\colon Y \to Z$ are multivalued mappings, then the *composition* of q and p is the multivalued mapping $q \circ p\colon X \to Z$ such that $(q \circ p)(x) = q(p(x))$ for all $x \in X$.

We always use adjectives such as "compact-valued", "finite-valued" etc. for multivalued mappings; the word "mapping" (or "function") without such an adjective will always mean a usual single-valued mapping (which we naturally identify with the corresponding singleton-valued mappings).

A multivalued mapping $p\colon X \to Y$ is called *upper semicontinuous* (or *usc*) if for every open set V in Y, the set $\{ x \in X : p(x) \subset V \}$ is open in X. It is easy to see that continuous functions and inverses of perfect mappings are compact-valued usc. If F is a closed subspace of X, then the mapping $p_F\colon X \to F$ defined by

$$p_F(x) = \begin{cases} \{x\} & \text{if } x \in F \\ \emptyset & \text{if } x \notin F \end{cases}$$

(the inverse of the embedding $i_F\colon F \hookrightarrow X$) is usc. A straightforward verification shows that a composition of compact-valued usc mappings is compact-valued and usc; a standard argument using the closedness of the graph of p in $X \times \beta Y$ proves the following:

Proposition 1.1. *Let $p\colon X \to Y$ be a multivalued mapping. Then the following conditions are equivalent:*

(a) *p is compact-valued usc.*
(b) *p is a composition of the inverse of a perfect mapping onto a closed subspace of X and a continuous function.*
(c) *There are a compact space K, a closed subspace F of $X \times K$ and a continuous function $f\colon F \to Y$ such that $p = f \circ i_F^{-1} \circ \pi_X^{-1}$, where $\pi_X\colon X \times K \to X$ is the projection and $i_F\colon F \to X \times K$ is the embedding.*

A family of sets $\mathcal{N}$ is called a *network with respect to a cover $\mathcal{C}$* of a space X if for every set $C \in C$ and every neighborhood U of C there is an element N of $\mathcal{N}$ such that $C \subset N \subset U$ [9]. Note that if $\mathcal{C}$ is a compact cover of X (that is, all elements of $\mathcal{C}$ are compact), and $\mathcal{N}$ is a network with respect to $\mathcal{C}$, then the family of the closures of the elements of $\mathcal{N}$ is also a network with respect to $\mathcal{C}$.

The next proposition sums up several well-known characterizations of Lindelöf Σ-spaces [2], [12].

Proposition 1.2. *Let X be a space. The following conditions are equivalent:*

(a) *X is a Lindelöf Σ-space.*
(b) *There are a compact cover $\mathcal{C}$ of X and a countable network $\mathcal{N}$ with respect to $\mathcal{C}$.*
(c) *There are a second-countable space M and a compact-valued usc mapping $p\colon M \to X$ such that $p(M) = X$.*
(d) *There are a second-countable space M, a space L and mappings $g\colon L \to M$ and $f\colon L \to X$ such that g is perfect and f is continuous onto.*
(e) *There are a second-countable space M, a compact space K, a closed subspace F of $M \times K$ and a continuous mapping $f\colon F \to X$ such that $f(F) = X$.*

The equivalence of (a) and (b) is immediate from the definition (see [9]). The equivalence of (c), (d) and (e) follows from Proposition 1.1. If $p\colon M \to X$ is a compact-valued usc mapping onto X, and $\mathcal{B}$ is a countable base for M, then $\{\, p(B) : B \in \mathcal{B} \,\}$ is a countable network with respect to the compact cover $\{\, p(m) : m \in M \,\}$ of the space X, so (c) implies (b). To verify that (b) implies (c), equip $\mathcal{N}$ with the discrete topology, and let M be the subspace of $\mathcal{N}^\omega$ (equipped with the product topology) consisting of all functions $m : \omega \to \mathcal{N}$ with the property that $\{\, m(i) : i \in \omega \,\} = \{\, N \in \mathcal{N} : C \subset N \,\}$ for some $C \in \mathcal{C}$. Then M is a second-countable space; the mapping $p\colon M \to X$ defined by the rule

$$p(m) = \bigcap \{\, m(i) : i \in \omega \,\}$$

is compact-valued, usc, and onto X.

Note that the cardinality of the cover $\mathcal{C}$ as in (b) cannot exceed 2^ω.

Of course, all compact spaces and all spaces with a countable network are Lindelöf Σ-spaces. From Proposition 1.2 it follows easily that the class of Lindelöf Σ-spaces is invariant with respect to images under compact-valued usc mappings (in particular, continuous images, closed subspaces and perfect preimages), countable products and countable unions.

2. The classes $\mathrm{L}\Sigma(\leqslant \kappa)$ and $\mathrm{KL}\Sigma(\leqslant \kappa)$

In this article we consider subclasses of the class of all Lindelöf Σ-spaces obtained by requiring that the elements of the compact cover $\mathcal{C}$ as in Proposition 1.2(b) have a given property. This leads to the following definition.

Definition 2.1. Let $\mathcal{K}$ be a class of compact spaces. Define $\mathrm{L}\Sigma(\mathcal{K})$ as the class of all spaces such that there are a second-countable space M and a compact-valued usc mapping $p\colon M \to X$ such that $p(M) = X$ and $p(m) \in \mathcal{K}$ for all $m \in M$.
We also define the class $\mathrm{KL}\Sigma(\mathcal{K})$ as the class of all spaces such that there are a *compact* second-countable space M and a compact-valued usc mapping $p\colon M \to X$ such that $p(M) = X$ and $p(m) \in \mathcal{K}$ for all $m \in M$.

Clearly, always $\mathrm{KL}\Sigma(\mathcal{K}) \subseteq \mathrm{L}\Sigma(\mathcal{K})$ and all spaces in $\mathrm{KL}\Sigma(\mathcal{K})$ are compact.

An argument similar to the proof of Proposition 1.2 gives the following:

Proposition 2.2. *Let X be a space and $\mathcal{K}$ a class of compact spaces. Then the following conditions are equivalent:*

- (a) $X \in \mathrm{L}\Sigma(\mathcal{K})$.
- (b) *There are a compact cover $\mathcal{C}$ of X such that $\mathcal{C} \subset \mathcal{K}$ and a countable network $\mathcal{N}$ with respect to $\mathcal{C}$.*

If the class $\mathcal{K}$ is closed with respect to continuous images and closed subspaces, then these conditions are also equivalent to

- (c) *There are a second-countable space M, a space L and mappings $g\colon L \to M$ and $f\colon L \to X$ such that g is perfect, f is continuous, and $g^{-1}(m) \in \mathcal{K}$ for all $m \in M$.*

(*d*) *There are a second-countable space M, a compact space K, a closed subspace F of $M \times K$ and a continuous mapping $f\colon F \to X$ such that $f(F) = X$ and $F \cap \pi_M^{-1}(m) \in \mathcal{K}$ for all $m \in M$, where $\pi_M\colon M \times K \to M$ is the projection.*

Similarly,

Proposition 2.3. *Let X be a space and $\mathcal{K}$ a class of compact spaces is invariant under continuous images and closed subspaces. Then the following conditions are equivalent:*

(*a*) $X \in \mathrm{KL\Sigma}(\mathcal{K})$.
(*b*) *There are a compact second-countable space M, a space L and mappings $g\colon L \to M$ and $f\colon L \to X$ such that g is perfect, f is continuous, and $g^{-1}(m) \in \mathcal{K}$ for all $m \in M$.*
(*c*) *There are a compact second-countable space M, a compact space K, a closed subspace F of $M \times K$ and a continuous mapping $f\colon F \to X$ such that $f(F) = X$ and $F \cap \pi_M^{-1}(m) \in \mathcal{K}$ for all $m \in M$.*

Thus, a compact space X is in $\mathrm{L\Sigma}(\mathcal{K})$ if and only if X has a countable closed cover $\mathcal{N}$ such that for every $x \in X$ the set $\bigcap\{\, N \in \mathcal{N} : x \in N \,\}$ belongs to $\mathcal{K}$. A (not necessarily compact) space X satisfying this condition is called *$\mathcal{K}$-approximable* in [13] and *weakly $\mathcal{K}$-fibered* in [14] (in fact, [14] deals only with the class of *weakly metrizably fibered* spaces which is the class of weakly $\mathcal{K}$-fibered spaces with $\mathcal{K}$ the class of all metrizable compacta). Note that a countably compact space which is $\mathcal{K}$-approximable is in $L\Sigma(\mathcal{K})$.

If κ is a cardinal, finite or infinite, we denote by $\mathrm{L\Sigma}(\leqslant \kappa)$ and $\mathrm{KL\Sigma}(\leqslant \kappa)$ the classes $\mathrm{L\Sigma}(\mathcal{K})$ and $\mathrm{KL\Sigma}(\mathcal{K})$ where $\mathcal{K}$ is the class of all compact spaces of weight $\leqslant \kappa$. Similarly, $\mathrm{L\Sigma}(< \kappa)$ and $\mathrm{KL\Sigma}(< \kappa)$ are the classes $\mathrm{L\Sigma}(\mathcal{K})$ and $\mathrm{KL\Sigma}(\mathcal{K})$ where $\mathcal{K}$ is the class of all compact spaces of weight $< \kappa$; let $\mathrm{L\Sigma}(\kappa) = \mathrm{L\Sigma}(\leqslant \kappa) \setminus \mathrm{L\Sigma}(< \kappa)$. Since the cardinality of a compact cover with respect to which there is a countable network does not exceed 2^ω, all spaces in $\mathrm{L\Sigma}(\leqslant \kappa)$ have cardinality at most $2^{\kappa+\omega}$, and if $\kappa \geqslant 2^\omega$, then the class $L\Sigma(\leqslant \kappa)$ coincides with the class of all Lindelöf Σ-spaces of network weight $\leqslant \kappa$.
When κ is a finite cardinal, "$\leqslant \kappa$" means "at most κ-element sets". Thus, $X \in \mathrm{L\Sigma}(n)$, $n \in \omega$, if X has a cover $\mathcal{C}$ consisting of at most n-element sets which has a countable network in X, but X does not have such a cover consisting of at most $(n-1)$-element sets. Obviously, $\mathrm{L\Sigma}(\leqslant 1)$ is the class of all spaces of countable network weight, and $\mathrm{KL\Sigma}(\leqslant 1)$ is the class of all metrizable compacta.
¿From Propositions 2.2 and 2.3 readily follows

Proposition 2.4. *Let κ be a cardinal. Then the classes $\mathrm{L\Sigma}(\leqslant \kappa)$, $\mathrm{L\Sigma}(< \kappa)$, $\mathrm{KL\Sigma}(\leqslant \kappa)$ and $\mathrm{KL\Sigma}(< \kappa)$ are invariant with respect to closed subspaces, continuous images and finite unions. The classes $\mathrm{L\Sigma}(\leqslant \kappa)$ and $\mathrm{L\Sigma}(< \kappa)$ are invariant with respect to countable unions.*

Since the product of a family of compact-valued usc mappings is compact-valued and usc, we have

Proposition 2.5. *If $X \in \mathrm{L\Sigma}(\leqslant \kappa)$ and $Y \in \mathrm{L\Sigma}(\leqslant \lambda)$, then $X \times Y \in \mathrm{L\Sigma}(\leqslant \lambda \cdot \kappa)$.*

and

Proposition 2.6. *If* $\{X_n : n \in \omega\}$ *is a countable family of spaces, and* $X_n \in \mathrm{L}\Sigma(\leqslant \kappa_n)$, $n \in \omega$, *then* $\prod\{X_n : n \in \omega\} \in \mathrm{L}\Sigma(\leqslant \kappa)$ *where* $\kappa = |A| \cdot \sup\{\kappa_n : n \in \omega\}$.

Examples 2.7. 1. The double arrow space is in $\mathrm{KL}\Sigma(2)$. Indeed, it admits a 2-to-1 perfect mapping onto the closed interval, and therefore is in $\mathrm{KL}\Sigma(\leqslant 2)$. It is not in $\mathrm{KL}\Sigma(1)$, because it has no countable network.

2. Let T be the unit circle, and $AD(T)$ its Alexandroff duplicate (see e.g. [5]). Then $AD(T)$ is in $\mathrm{KL}\Sigma(2)$, because $AD(T)$ is not metrizable and admits a perfect 2-to-1 mapping onto T.

3. The space $A(2^\omega)$ is a non-metrizable continuous image of $AD(T)$, and hence $A(2^\omega) \in \mathrm{L}\Sigma(2)$. Therefore, $A(\kappa) \in \mathrm{L}\Sigma(\leqslant \omega)$ iff $A(\kappa) \in \mathrm{L}\Sigma(\leqslant 2)$ iff $\kappa \leqslant 2^\omega$.

A compact space X is called *metrizably fibered* [14] if X admits a continuous mapping with metrizable fibers onto a metrizable space. Clearly, all metrizably fibered spaces are in $\mathrm{KL}\Sigma(\leqslant \omega)$, and by Proposition 2.3 every space in $\mathrm{KL}\Sigma(\leqslant \omega)$ is a continuous image of a metrizably fibered compact space. Since all metrizably fibered compact spaces are first-countable, $A(\omega_1)$ is in $\mathrm{KL}\Sigma(2)$, but is not metrizably fibered. Note also that every space in $\mathrm{L}\Sigma(\leqslant \omega)$ is weakly metrizably fibered, and that a compact space is weakly metrizably fibered if and only if it is in $\mathrm{L}\Sigma(\leqslant \omega)$ (the family of all finite intersections of members of the family $\mathcal{N}$ from the definition of weakly $\mathcal{K}$-metrizable spaces cited above is a network with respect to the cover of X formed by the sets $C_x = \bigcap\{N \in \mathcal{N} : x \in N\}$, $x \in X$). It is shown in [6] that every such compact space is sequential.
Since every metrizably fibered compact space is first-countable, it follows that every space in $\mathrm{KL}\Sigma(\leqslant \omega)$ is Fréchet.

Example 2.8. (Example 2.13 in [14]). Let K be the one-point compactification of a Mrówka space. Then K is a countable union of subspaces in $\mathrm{L}\Sigma(\leqslant 2)$, (countably many singletons and the one-point compactification of a discrete space of cardinality $\leqslant 2^\omega$) and hence is itself in $\mathrm{L}\Sigma(\leqslant 2)$. Since K is not Fréchet, it is not in $\mathrm{KL}\Sigma(\leqslant \omega)$.

As we mentioned above, every compact space in $\mathrm{L}\Sigma(\leqslant \omega)$ is sequential [6], and therefore has countable tightness [14]. Of course, every space in $\mathrm{L}\Sigma(\leqslant 1)$ has countable tightness. The next example shows that not all spaces in $\mathrm{L}\Sigma(< \omega)$ have countable tightness.

Example 2.9. Let X be the subspace of 2^{ω_1} which is the union of the set S of all points that have finitely many coordinates equal to 1 and the singleton $\{\mathbf{1}\}$ where $\mathbf{1}$ is the point whose all coordinates are equal to 1. It is easy to see that the tightness of X at the point $\mathbf{1}$ is uncountable. From Lemma 2.9 in [10] it follows that S is a countable union of continuous images of spaces of the form $A(\omega_1)^n \times 2^n$, $n \in \omega$, and hence is in the class $\mathrm{L}\Sigma(< \omega)$. Thus, X is a σ-compact space in $\mathrm{L}\Sigma(< \omega)$ of uncountable tightness.

Question 2.10. Does the class $\mathrm{L}\Sigma(2)$ contain a space of uncountable tightness? Does any of the classes $\mathrm{L}\Sigma(n)$, $n \in \omega$, contain a space of uncountable tightness?

Recall that a *free sequence of length* κ in a topological space X is a function $f\colon \kappa \to X$ such that for every $\alpha < \kappa$, the sets $\{\, f(\beta) : \beta < \alpha \,\}$ and $\{\, f(\beta) : \alpha \leqslant \beta < \kappa \,\}$ have disjoint closures. If X is a compact space, then the tightness of X is equal to the supremum of the lengths of free sequences in X [1].

Theorem 2.11. *Assume that κ is an uncountable regular cardinal, and $X \in \mathrm{L}\Sigma(<\kappa)$. Then every free sequence in X has length $<\kappa$.*

Proof. Fix a compact cover $\mathcal{C}$ with a countable network $\mathcal{N}$ in X so that every element of $\mathcal{C}$ has weight $<\kappa$.

Suppose $f\colon \kappa \to X$ is a free sequence in X. For every $\alpha < \beta \leqslant \kappa$ put $F(\alpha,\beta) = \operatorname{cl} f[[\alpha,\beta)]$. Then by the definition of a free sequence $F(0,\alpha) \cap F(\alpha,\kappa) = \emptyset$.

Since $|\mathcal{N}| = \omega$, there exists $\delta < \kappa$ such that for every $N \in \mathcal{N}$ either $\sup f^{-1}[N] < \delta$ or $f^{-1}[N]$ is unbounded in κ. Fix $C \in \mathcal{C}$ so that $f(\delta) \in C$. We claim that there exists $\alpha_0 \in [\delta,\kappa)$ such that $C \cap F(\alpha_0,\beta) = \emptyset$ for every $\beta > \alpha_0$. Indeed, otherwise for every $\alpha > \delta$ pick $p_\alpha \in C$ such that $p_\alpha \in F(\alpha,\varrho(\alpha))$ for some $\varrho(\alpha) > \alpha$. Choose unbounded $S \subseteq \kappa \setminus \delta$ such that $\varrho(\alpha) < \alpha'$ whenever $\alpha,\alpha' \in S$ and $\alpha < \alpha'$. Then $\{p_\alpha \colon \alpha \in S\}$ is a free sequence of length κ in C, which contradicts the assumption that $w(C) < \kappa$.

Fix $\alpha_0 > \delta$ so that $C \cap F(\alpha_0,\beta) = \emptyset$ for every $\beta > \alpha_0$. Find $\beta_0 > \alpha_0$ such that $N \cap f^{-1}[[\alpha_0,\beta_0)] \neq \emptyset$ whenever $f^{-1}[N]$ is unbounded in κ and $N \in \mathcal{N}$. Then $X \setminus F(\alpha_0,\beta_0)$ is a neighborhood of C. Thus, there exists $N \in \mathcal{N}$ such that $C \subseteq N$ and $N \cap F(\alpha_0,\beta_0) = \emptyset$. Then $f^{-1}[N]$ is bounded in κ and therefore $\sup f^{-1}[N] < \delta < \alpha_0$. On the other hand $f(\delta) \in N$, a contradiction. □

Corollary 2.12. *Assume $X \in \mathrm{L}\Sigma(\leqslant\kappa)$ is compact and $\kappa \geqslant \omega$. Then $\mathrm{t}(X) \leqslant \kappa$.*

Remark 2.13. The same argument proves the following statement: *Assume that κ is an uncountable regular cardinal, and $X \in \mathrm{L}\Sigma(t<\kappa)$. Then every free sequence in X has length $<\kappa$*; here "$t < \kappa$" is the class of all compact spaces of tightness $<\kappa$. A similar statement for countably compact spaces was proved by Tkačenko [13, Assertion 2.2]. A special case (for compact spaces) was proved by Tkachuk [13, Thm. 2.11]

Gerlits and Szentmiklóssy proved in [6] that the Helly space belongs to $\mathrm{L}\Sigma(\leqslant\omega)$ (in fact they showed that the Helly space can be mapped onto a metric space by a map with metrizable fibers – this is a stronger property than being in $\mathrm{L}\Sigma(\leqslant\omega)$). Todorčević proved in [17] a dichotomy for Rosenthal compact spaces, where one of the assertions is "being a two-to-one preimage of a compact metric space". In view of the next result, it is natural to ask whether all Rosenthal compacta belong to $\mathrm{L}\Sigma(\leqslant\omega)$.

Proposition 2.14. *For every Polish space X, the space*

$$DC_\omega(X) = \{f \in \mathbb{R}^X : f \text{ has only countably many points of discontinuity }\}$$

is metrizably-approximable. In particular, every compact subspace of $DC_\omega(X)$ belongs to $\mathrm{L}\Sigma(\leqslant\omega)$.

Proof. Fix a countable base $\mathcal{B}$ in X. Define

$$N_{u,J} = \{f \in DC_\omega(X) \colon f[\operatorname{cl} u] \subseteq J\}$$

and let $\mathcal{N} = \{N_{u,J} \colon u \in \mathcal{B},\ J \text{ is a closed rational interval}\}$. Then $\mathcal{N}$ is a countable family of closed subsets of $DC_\omega(X)$.

Let $C_f = \bigcap\{N \in \mathcal{N} \colon f \in N\}$. We claim that C_f is metrizable for every $f \in DC_\omega(X)$.

Fix $f \in DC_\omega(X)$ and $g \in C_f$. We have $g(x) = f(x)$ whenever f is continuous at x. Indeed, if $f(x) \neq g(x)$, then we can find $u \in \mathcal{B}$ and a closed rational interval J such that $x \in u$, $f[\operatorname{cl} u] \subseteq J$ and $g(x) \notin J$. Then $f \in N_{u,J} \in \mathcal{N}$ and $g \notin N_{u,J}$, a contradiction.

Let A denote the set of all points of discontinuity of f. We have proved that

$$C_f \subseteq \{h \in \mathbb{R}^X \colon f \restriction (X \setminus A) = h \restriction (X \setminus A)\}.$$

The set on the right-hand side is homeomorphic to $\mathbb{R}^A$ with the product topology. It follows that C_f is metrizable, because $|A| \leqslant \omega$. □

3. Small Corson compacta need not be in $\mathrm{L}\Sigma(\leqslant \omega)$

Tkachuk proved in [14] that every Eberlein compact space of weight at most continuum (equivalently, of size at most continuum) belongs to $\mathrm{L}\Sigma(\leqslant \omega)$. This is not true for Corson compacta. A consistent counterexample is given in [14]. We show that a certain Corson compact space constructed (in ZFC) by Todorčević in [15] (see also [16, p. 287]) does not belong to $\mathrm{L}\Sigma(\leqslant \omega)$. This answers Tkachuk's question from [14].

We recall the construction of Todorčević's Corson compact space. Fix a stationary co-stationary set $A \subseteq \omega_1$ and let $T(A)$ be the collection of all subsets of A which are closed in ω_1. Then $\langle T(A), \subseteq \rangle$ is a tree of height ω_1 whose all branches are countable. Let $Y(A)$ denote the collection of all initial branches of the tree $T(A)$ (an *initial branch* is a linearly ordered subset which is also closed downwards). Then $Y(A)$ is a compact subspace of the Cantor cube $\mathcal{P}(T(A))$. Since all branches of $T(A)$ are countable, $Y(A)$ is Corson compact.

Note that $|T(A)| \leqslant {\omega_1}^\omega = 2^\omega$ and therefore $Y(A)$ is a "small" Corson compact space.

Proposition 3.1. *For every stationary co-stationary set $A \subseteq \omega_1$, $Y(A) \notin \mathrm{L}\Sigma(\leqslant \omega)$.*

Proof. Suppose $Y(A) \in \mathrm{L}\Sigma(\leqslant \omega)$ and let $\mathcal{C}$ be a cover of $Y(A)$ consisting of metric compacta and $\mathcal{N}$ a countable network with respect to $\mathcal{C}$. It has been proved in [16, p. 287] that $T(A)$ is a Baire partial order, which means that, as a forcing notion, it does not add new countable sequences.

Assume now that we are working in a countable transitive ZFC model M and let G be a $T(A)$-generic filter over M. In $M[G]$, the space $Y(A)$ (with the topology generated by open sets from the ground model) is still in $\mathrm{L}\Sigma(\leqslant \omega)$, because $\mathcal{C}$ still consists of metric compacta (by the fact that there are no new sequences in $M[G]$), and $\mathcal{N}$ is still a countable network for $\mathcal{C}$.

On the other hand, the generic filter G introduces an uncountable strictly decreasing chain of open subsets of $Y(A)$. Thus, $Y(A)^{M[G]}$ contains an uncountable free sequence, which contradicts Theorem 2.11. □

4. Some results concerning classes $\mathrm{L}\Sigma(\leqslant n)$

Proposition 4.1. *Let $n \in \omega$ and assume X is a space which has a disjoint family of open sets $\{U_\alpha : \alpha < \omega_1\}$, and for each $\alpha < \omega_1$ there is a closed set $Y_\alpha \subset U_\alpha$ such that $Y_\alpha \notin \mathrm{L}\Sigma(\leqslant n)$. Then $X \notin \mathrm{L}\Sigma(\leqslant n+1)$.*

Proof. Suppose $X \in \mathrm{L}\Sigma(\leqslant n+1)$ and fix a cover $\mathcal{C} \subseteq [X]^{\leqslant n+1}$ which has a countable network $\mathcal{N}$. For each $\alpha < \omega_1$ the collection $\{Y_\alpha \cap C : C \in \mathcal{C}\}$ is a cover of Y_α with a countable network in Y_α. Since $Y_\alpha \notin \mathrm{L}\Sigma(\leqslant n)$, there exists $C_\alpha \in \mathcal{C}$ such that $C_\alpha \subseteq Y_\alpha$. Choose $N_\alpha \in \mathcal{N}$ so that $C_\alpha \subset N_\alpha \subset U_\alpha$. Then $N_\alpha \neq N_\beta$ for $\alpha \neq \beta$, a contradiction. □

Proposition 4.2. *Let $n \in \omega$ and assume $\{X_\xi : \xi < \kappa\} \subseteq \mathrm{L}\Sigma(\leqslant n)$ is a family of compact spaces and $\kappa \leqslant 2^\omega$. Let X be the one-point compactification of $\bigoplus_{\xi<\kappa} X_\xi$. Then $X \in \mathrm{L}\Sigma(\leqslant n+1)$. If $X_\xi \in \mathrm{L}\Sigma(n)$ for uncountably many ξ, then $X \in \mathrm{L}\Sigma(n+1)$.*

Proof. For each X_ξ fix a cover $\mathcal{C}_\xi \subseteq [X_\xi]^{\leqslant n}$ with a countable network $\mathcal{N}_\xi$ in X_ξ. We assume that $X_\xi \cap X_\eta = \emptyset$ whenever $\xi \neq \eta$ and that $X = \{\infty\} \cup \bigcup_{\xi<\kappa} X_\xi$.
Define $\mathcal{C} = \{C \cup \{\infty\} : C \in C_\xi,\ \xi < \kappa\}$. We will show that $\mathcal{C}$ has a countable network in X.
Let $\mathcal{N}_\xi = \{N^\xi_k : k < \omega\}$ and fix a countable family $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ which separates finite sets (here we use the fact that $\kappa \leqslant 2^\omega$). Let $A^* = \bigcup_{\xi \in A} X_\xi$. Define

$$\mathcal{N} = \Big\{\{\infty\} \cup \Big(A^* \cap \bigcup_{\xi<\kappa} N^\xi_k\Big) : k < \omega,\ A \in \mathcal{A}\Big\}.$$

Then $\mathcal{N}$ is a countable family. We claim that $\mathcal{N}$ is a network for $\mathcal{C}$. Fix $C = C_0 \cup \{\infty\}$, where $C_0 \in \mathcal{C}_\xi$. Fix an open set $U \subseteq X$ such that $C \subseteq U$. Then U is a neighborhood of ∞, so the set $F = \{\eta < \kappa : X_\eta \not\subseteq U\}$ is finite. Find $A \in \mathcal{A}$ such that $\xi \in A$ and $(F \setminus \{\xi\}) \cap A = \emptyset$. Find $k < \omega$ such that $C_0 \subseteq N^\xi_k \subseteq U \cap X_\xi$. Let $M = \{\infty\} \cup (A^* \cap \bigcup_{\eta<\kappa} N^\eta_k)$. Then $M \in \mathcal{N}$ and $C \subseteq M \subseteq U$.
The second statement follows from Proposition 4.1. □

Corollary 4.3. *All classes $\mathrm{L}\Sigma(n)$, for $n < \omega$, as well as $\mathrm{L}\Sigma(\omega)$, restricted to compact spaces, are nonempty. In fact, for every $n \in \omega$ there exists a scattered compact space X_n of height n and of cardinality ω_1, such that $X_n \in \mathrm{L}\Sigma(n+1)$.* □

By Proposition 2.5,

Proposition 4.4. *Assume $X \in \mathrm{L}\Sigma(\leqslant n)$ and $Y \in \mathrm{L}\Sigma(\leqslant k)$, where $n, k < \omega$. Then $X \times Y \in \mathrm{L}\Sigma(\leqslant nk)$.* □

Proposition 4.2 implies that $A(\kappa) \in \mathrm{L}\Sigma(2)$ if $\omega < \kappa \leqslant 2^\omega$. Below we show that $A(\omega_1)^n \in \mathrm{L}\Sigma(n+1)$, but $A(\omega_1) \times A(\omega_2) \notin \mathrm{L}\Sigma(3)$ even if $\omega_2 \leqslant 2^\omega$.

Theorem 4.5. $A(\omega_1)^n \in \mathrm{L}\Sigma(n+1)$.

Proof. Let $A(\omega_1) = \omega_1 + 1$, where all points of ω_1 are isolated. The proof is by induction on n. The case $n = 1$ is proved above. Suppose that $n > 1$. Since $A(\omega_1)^n$ contains ω_1 disjoint clopen copies of $A(\omega_1)^{n-1}$, $A(\omega_1)^n \notin \mathrm{L}\Sigma(< n+1)$ by Proposition 4.1.

For each permutation $p : n \to n$, consider the subset X_p of $A(\omega_1)^n$ defined by

$$X_p = \{(x_i)_{i<n} : x_{p(i)} \leq x_{p(i+1)} \text{ for all } i < n-1\}.$$

Then $A(\omega_1)^n = \bigcup\{X_p : p \in {}^n n \text{ is a permutation}\}$. By Proposition 2.4, it suffices to prove that each $X_p \in \mathrm{L}\Sigma(\leq n+1)$. Since all X_p are homeomorphic, it suffices to prove that $X_{id} \in \mathrm{L}\Sigma(\leq n+1)$. We first define a cover by $n+1$ element sets:

For each $\overline{\alpha} = (\alpha_0, \alpha_1, ..., \alpha_{n-1}) \in X_{id} \cap \omega_1^n$ and for each $k < n$ let $\overline{\alpha}^k$ be the element of X_{id} obtained by changing the last $n-k$ coordinates of $\overline{\alpha}$ to ω_1. So, $\overline{\alpha}^n = \overline{\alpha}$ and $\overline{\alpha}^0 = (\omega_1, \omega_1, ..., \omega_1)$. Let

$$C_{\overline{\alpha}} = \{\overline{\alpha}^k : k \leq n,\ \overline{\alpha} \in X_p \cap \omega_1^n\}.$$

These sets form our cover of X_{id} by $n+1$-element sets.

We now define a countable network for this family in X_{id}. The elements of this network will be constructed from three countable families of sets:

Family 1: We assume, by induction, that the collection of similarly defined sets in $A(\omega_1)^{n-1}$ has a countable network (note that for $n = 1$ this follows from 2.7.3). Since $A(\omega_1)^{n-1}$ is naturally identified with the subspace $Y = A(\omega_1)^{n-1} \times \{\omega_1\}$, there is a countable network for the sets $C_{\overline{\alpha}} \cap Y$ contained in $Y \cap X_{id}$. Call this countable family $\mathcal{N}_Y$. Notice that for each $\overline{\alpha} \in X_{id}$, the only point of $C_{\overline{\alpha}}$ not contained in Y is $\overline{\alpha}$.

Family 2: Fix a countable family $\mathcal{F} \subseteq {}^{\omega_1}\omega_1$ of functions such that $\{(\alpha, \beta) : \beta \leq \alpha\} = \bigcup \mathcal{F}$. For a sequence $\overline{f} = (f_i : i < n-1)$ in $\mathcal{F}$, let $N_{\overline{f}} \subseteq \omega_1^n$ be the set of $(\alpha_i)_{i<n}$ such that $\alpha_i = f_i(\alpha_{i+1})$ for each $i < n-1$. Notice that each $N_{\overline{f}} \subseteq X_{id}$.

Family 3: Let $\mathcal{N}_0$ be a countable family of subsets of ω_1 that separates points from finite sets.

Our network will consist of sets of the following form:

$$N \cup \big(N_{\overline{f}} \cap \prod_{i<n} N_i\big)$$

Where $N \in \mathcal{N}_Y$, $\overline{f} \in \mathcal{F}^{n-1}$ and each $N_i \in \mathcal{N}_0$.

To verify that this family is a network with respect to the cover by the sets $C_{\overline{\alpha}}$, fix $\overline{\alpha} \in X_{\mathrm{id}}$ and an open set $U \supseteq C_{\overline{\alpha}}$. By construction, there is an $N \in \mathcal{N}_Y$, such that $C_{\overline{\alpha}} \setminus \{\overline{\alpha}\} \subseteq N \subseteq U$. Since $\overline{\alpha}_0 = (\omega_1, \omega_1, ..., \omega_1) \in U$, we may fix finite sets $F_i \subseteq \omega_1$ for $i < n$ so that

$$V(0) = \prod_{i<n} (\omega_1 \setminus F_i) \subseteq U.$$

Similarly, for each $0 < k \leq n$, we may fix a basic open set $V(k)$ containing $\overline{\alpha}_k$. Without loss of generality, we may assume that

$$V(k) = \{\alpha_0\} \times \cdots \times \{\alpha_{k-1}\} \times \prod_{k \leq i < n} (\omega_1 \setminus F_i) \subseteq U.$$

Moreover, we may assume that $\alpha_i \in F_i$ for each $i < n$.

For each $i < n$ fix $N_i \in \mathcal{N}_0$ such that $N_i \cap F_i = \{\alpha_i\}$ (that is, N_i separates the point α_i from the finite set $F_i \setminus \{\alpha_i\}$).

Next, for each $i < n-1$, fix $f_i \in \mathcal{F}$ such that $f_i(\alpha_i + 1) = \alpha_i$. Clearly,

$$\overline{\alpha} \in N_{(f_i)} \cap \prod_{i<n} N_i,$$

so it suffices to prove that $N_{(f_i)} \cap \prod_{i<n} N_i \subseteq U$. So suppose not, and fix $\overline{\beta} = (\beta_0, \beta_1, ..., \beta_{n-1}) \in (N_{(f_i)} \cap \prod_{i<n} N_i) \setminus U$. Fix k maximal such that $\beta_k = \alpha_k$ (if there is no such k, then $\beta_k \notin F_k$ for every k and hence $\overline{\beta} \in V(0) \subseteq U$). Then $\beta_i = \alpha_i$ for all $i \leq k$, since $\overline{\beta} \in N_{(f_i)}$, and $\beta_i \notin F_i$ for all $i > k$. Thus, $\overline{\beta} \in V(k) \subseteq U$, a contradiction. $\square$

Let us now prove that $A(\omega_2) \times A(\omega_2) \notin \mathrm{L}\Sigma(3)$.

Given two families $\mathcal{C}_1, \mathcal{C}_2$ of subsets of a set X, denote $\mathcal{C}_1 \wedge \mathcal{C}_2 = \{\, C_1 \cap C_2 : C_1 \in \mathcal{C}_1,\ C_2 \in \mathcal{C}_2 \,\}$.

Lemma 4.6. *Let $\mathcal{C}_1$, $\mathcal{C}_2$ be covers of a space X by finite sets, and $\mathcal{N}_1$, $\mathcal{N}_2$ networks with respect to $\mathcal{C}_1$ and $\mathcal{C}_2$. Then $\mathcal{N}_1 \wedge \mathcal{N}_2$ is a network with respect to $\mathcal{C}_1 \wedge \mathcal{C}_2$.*

Proof. Let $C_1 = \{u_1, \ldots, u_k, s_1, \ldots, s_m\} \in \mathcal{C}_1$, $C_2 = \{u_1, \ldots, u_k, t_1, \ldots, t_l\} \in \mathcal{C}_2$, and let U be a neighborhood of $C = C_1 \cap C_2 = \{u_1, \ldots, u_k\}$ in X. Fix disjoint neighborhoods $U_{u_1}, \ldots, U_{t_l}$ of the points in $C_1 \cup C_2$ so that $U_{u_1} \cup \cdots \cup U_{u_k} \subset U$. Fix $N_1 \in \mathcal{N}_1$ and $N_2 \in \mathcal{N}_2$ so that

$$C_1 \subset N_1 \subset \bigcup\{\, U_{u_i} : 1 \leqslant i \leqslant k \,\} \cup \bigcup\{\, U_{s_i} : 1 \leqslant i \leqslant m \,\}$$

and

$$C_2 \subset N_2 \subset \bigcup\{\, U_{u_i} : 1 \leqslant i \leqslant k \,\} \cup \bigcup\{\, U_{t_i} : 1 \leqslant i \leqslant l \,\}$$

Then $N = N_1 \cap N_2$ is an element of $\mathcal{N}_1 \wedge \mathcal{N}_2$ such that $C_1 \cap C_2 \subset N \subset \bigcup\{\, U_{u_i} : 1 \leqslant i \leqslant k \,\} \subset U$. $\square$

Theorem 4.7. $A(\omega_2)^2 \notin \mathrm{L}\Sigma(3)$.

Proof. If $\omega_2 > 2^\omega$, then $A(\omega_2)$ is not in $\mathrm{L}\Sigma(3)$, because every space in $\mathrm{L}\Sigma(n)$ has cardinality $\leqslant 2^\omega$.

So assume $\omega_2 \leqslant 2^\omega$. Let $A(\omega_2) = \omega_2 + 1$, where all points of ω_2 are isolated. Let $\mathcal{C}_0$ be the cover of $A(\omega_2)^2$ by the sets of the form $C_{\alpha\beta} = \{(\alpha, \beta), (\alpha, \omega_2), (\omega_2, \beta), (\omega_2, \omega_2)\}$, $\alpha, \beta < \omega_2$. Then there is a countable network $\mathcal{N}_0$ with respect to $\mathcal{C}_0$ (see Example 2.7(3) and equivalence of (a) and (b) in Proposition 2.2).

Now suppose that there exists a cover $\mathcal{C}$ of $A(\omega_2)^2$ with at most 3-point sets and a countable network $\mathcal{N}$ with respect to $\mathcal{C}$. Replacing $\mathcal{C}$ with $\mathcal{C} \wedge \mathcal{C}_0$ and $\mathcal{N}$ with $\mathcal{N} \wedge \mathcal{N}_0$ if necessary, we may assume that every element of $\mathcal{C}$ is contained in a set of the form $C_{\alpha\beta}$.

We will say that an element C of $\mathcal{C}$ is of type 1 if for some α, β, $C \subset \{(\alpha, \beta), (\omega_2, \beta), (\omega_2, \omega_2)\}$, is of type 2 if $C \subset \{(\alpha, \beta), (\alpha, \omega_2), (\omega_2, \omega_2)\}$, and of type 3 if $C \subset \{(\alpha, \beta), (\alpha, \omega_2), (\omega_2, \beta)\}$. Note that the elements of $\mathcal{C}$ of the three types cover $\omega_2 \times \omega_2$. Furthermore, there are at most countably many elements of type 3. Indeed, otherwise the union P of all elements of $\mathcal{C}$ of type 3 would be a Lindelöf Σ-space, which is impossible, because one of the three sets

$$P \cap (A(\omega_2) \times \{\omega_2\}), \quad P \cap \big(\{\omega_2\} \times A(\omega_2)\big), \quad \{\, (\alpha, \beta) \in P : (\alpha, \omega_2) \notin P,\ (\omega_2, \beta) \notin P \,\}$$

would be an uncountable closed discrete subspace of P. Removing from $A(\omega_2)$ all elements of ω_2 that occur as coordinates of points of P, and taking intersections of $\mathcal{C}$ the square of the remaining set, we will obtain a cover without elements of type 3 and the intersection of $\mathcal{N}$ with this square is a countable network with respect to this new cover. Thus, we may assume that all elements of $\mathcal{C}$ are of type 1 or 2.

Let $(\alpha,\beta) \in \omega_2 \times \omega_2$; choose an element C of $\mathcal{C}$ such that $(\alpha,\beta) \in C$. If C is of type 1, then $V = (A(\omega_2) \times \{\beta\}) \cup (A(\omega_2) \setminus \{\alpha\})^2$ is a neighborhood of C, so there is an $N \in \mathcal{N}$ such that $C \subset N \subset V$. Obviously, $N \cap (\{\alpha\} \times \omega_2) = \{(\alpha,\beta)\}$. Similarly, if C is of type 2, there is an $N \in \mathcal{N}$ such that $N \cap (\omega_2 \times \{\beta\}) = \{(\alpha,\beta)\}$. Thus, the sets

$$B_1 = \{\, (\alpha,\beta) \in \omega_2 \times \omega_2 : N \cap (\{\alpha\} \times \omega_2) = \{(\alpha,\beta)\} \text{ for some } N \in \mathcal{N} \,\}$$

and

$$B_2 = \{\, (\alpha,\beta) \in \omega_2 \times \omega_2 : N \cap (\omega_2 \times \{\beta\}) = \{(\alpha,\beta)\} \text{ for some } N \in \mathcal{N} \,\}$$

cover $\omega_2 \times \omega_2$. Note that for any given $\alpha \in \omega_2$ and $N \in \mathcal{N}$ there is at most one $\beta \in \omega_2$ such that $N \cap (\{\alpha\} \times \omega_2) = \{(\alpha,\beta)\}$, so for every $\alpha \in \omega_2$, the set $B_1 \cap (\{\alpha\} \times \omega_2)$ is at most countable. Similarly, for every $\beta \in \omega_2$ the set $B_2 \cap (\omega_2 \times \{\beta\})$ is at most countable. The existence of such a pair of sets covering $\omega_2 \times \omega_2$ contradicts a theorem of Kuratowski [8]. □

Note that the same proof shows that $A(\omega_1) \times A(\omega_2) \notin L\Sigma(3)$.

It appears natural to expect that if $X \in \mathrm{L}\Sigma(n)$ for some $n > 1$, then the sequence n_k such that $X^k \in \mathrm{L}\Sigma(n_k)$ should increase to infinity; hence the following question:

Question 4.8. Suppose $X^\omega \in \mathrm{L}\Sigma(< \omega)$. Must X have a countable network?

The results in this section allow to obtain a consistently positive answer to this question.

Proposition 4.9. *Suppose $X \in \mathrm{L}\Sigma(< \omega)$. Then there are subspaces $X_n \subset X$, $n \in \omega$, such that $X_n \in \mathrm{L}\Sigma(\leqslant n)$, and $X = \bigcup\{\, X_n : n \in \omega \,\}$.*

Proof. Let $p\colon M \to X$ be a finite-valued usc mapping from a second-countable space M such that $p(M) = X$. For every $n \in \omega$, put $M_n = \{\, m \in M : |p(m)| \le n \,\}$ and $X_n = p(M_n)$. □

Proposition 4.10. *Assume X is a space such that $X^\omega \in \mathrm{L}\Sigma(< \omega)$. Then for some $n \in \omega$, $X^\omega \in \mathrm{L}\Sigma(n)$.*

Proof. For every $k \in \omega$, let $\pi_k\colon (X^\omega)^\omega \to X^\omega$ be the projection to the kth factor. Since $(X^\omega)^\omega$ is homeomorphic to X^ω, we have $(X^\omega)^\omega = \bigcup\{\, X_k : k \in \omega \,\}$ where $X_k \in \mathrm{L}\Sigma(\leqslant k)$. Obviously, $\pi_n(X_n) = X^\omega$ for some $n \in \omega$. Then $X^\omega \in \mathrm{L}\Sigma(\leqslant n)$. □

Theorem 4.11. *If $X^\omega \in \mathrm{L}\Sigma(< \omega)$, then X^ω is hereditarily separable.*

Proof. Let us first show that X^ω has no uncountable discrete subspaces. By Lemma 4.10, $X^\omega \in \mathrm{L}\Sigma(n)$ for some $n \in \omega$. Let D be a discrete subspace of X, and let F be its closure in X. Since F^ω is a closed subspace of X^ω, $F^\omega \in \mathrm{L}\Sigma(\leqslant n)$. If D were uncountable, this would be impossible, because from Proposition 4.1 by a simple inductive argument would follow $F^n \notin \mathrm{L}\Sigma(\leqslant n)$. Thus, every discrete subspace of X is countable.

Since X^ω is homeomorphic to its square, it follows that X^ω is hereditarily separable or hereditarily Lindelöf [20, Theorem 1]. If X^ω is hereditarily Lindelöf, then it has G_δ-diagonal, and since it is a Lindelöf Σ-space, it must have countable network [3]. $\square$

Thus, if the answer to Question 4.8 is negative, then there is a strong S-space. Since MA_{ω_1} implies that there are no strong S-spaces [7], we get

Corollary 4.12. *If MA_{ω_1} holds, and $X^\omega \in \mathrm{L}\Sigma(<\omega)$, then X has countable network.*

5. Dense metrizable subspaces

Tkačenko asks in [13] whether every compact space in $\mathrm{L}\Sigma(\leqslant\omega)$ has a dense metrizable subspace. One cannot hope for a completely metrizable dense subspace because, for example, all metrizable subspaces of the double arrow space (which is in $\mathrm{KL}\Sigma(\leqslant 2)$) are countable.

We give a partial answer to Tkačenko's question, namely, we show that every space in $\mathrm{KL}\Sigma(<\omega)$ has a dense metrizable subspace. On the other hand, under $\neg\,\mathrm{MA}_{\omega_1}$, there exists a non-separable ccc space $X \in \mathrm{KL}\Sigma(\leqslant\omega)$ which is even metrizably fibered; see Theorem 3.5 in the survey article of Todorčević [18]. Observe that X cannot have a dense metrizable subspace, because such a subspace would be ccc and therefore separable. This gives a consistent negative answer to Tkačenko's question (see Problems 3.1 and 3.5 in [13]).

Todorčević proved in [17] that every Rosenthal compact space has a dense metrizable subspace. The key fact in the proof is the absoluteness of the class of Rosenthal compacta with respect to forcing extensions. We use the same idea, proving that the class $\mathrm{KL}\Sigma(<\omega)$ is absolute.

The following result is due to H.E. White, Jr. [19].

Proposition 5.1 (White Jr. [19])**.** *Every first countable Hausdorff space with a σ-disjoint π-base contains a dense metrizable subspace.*

Let X be a compact space and let $\mathbb{L} = \mathrm{Closed}(X)$, the collection of all closed subsets of X or let $\mathbb{L}$ be a sublattice of $\mathrm{Closed}(X)$ which is at the same time a closed base (we shall say that $\mathbb{L}$ is a *basic lattice* for X). Observe that every point of X corresponds to an ultrafilter (= a maximal filter) in $\mathbb{L}$. More precisely, the compact space X can be recovered from $\mathbb{L}$ as the space $\mathrm{Ult}(\mathbb{L})$ of all ultrafilters in $\mathbb{L}$; the topology is generated by closed sets of the form $a^+ = \{p \in \mathrm{Ult}(\mathbb{L})\colon a \in p\}$, where $a \in \mathbb{L}$. This gives an idea of how to interpret a compact space in a forcing extension. Namely, if X is a compact space in a ground model (a countable transitive model of a "large enough" part of ZFC) M and if G is a $\mathbb{P}$-generic filter over M, where $\mathbb{P} \in M$ is a fixed forcing notion, then we can define $X[G]$, the *interpretation of* X in $M[G]$, as $\mathrm{Ult}(\mathbb{L})$ computed in $M[G]$, where $\mathbb{L} = \mathrm{Closed}(X)$ (in fact, $\mathbb{L}$ can be any basic lattice for X). In $M[G]$, $\mathbb{L}$ is the same algebraic object, but $\mathrm{Ult}^{M[G]}(\mathbb{L})$ is usually different from $\mathrm{Ult}^M(\mathbb{L})$. It can be proved that the definition of $X[G]$ does not depend on the choice of $\mathbb{L}$ (as long as $\mathbb{L}$ is a basic lattice for X).

Let us remark that compact spaces in forcing extensions and absoluteness were already studied by Bandlow [4].

A lattice $\langle \mathbb{L}, +, \cdot, 0, 1\rangle$ is *normal* if it is distributive and for every $a, b \in \mathbb{L}$ such that $a \cdot b = 0$ there exist $a', b' \in \mathbb{L}$ such that

$$a \cdot b' = a' \cdot b = 0 \quad \text{and} \quad a' + b' = 1.$$

Every basic lattice for a compact space is normal.

The following result is due to Todorčević [17], although it is not stated explicitly in [17]. For the proof see Lemma 4 of [17].

Proposition 5.2. *Assume X is a compact space such that for every $\mathrm{RO}(X)$-generic filter G over some ground model containing X, the extension $X[G]$ has countable tightness. Then X has a σ-disjoint π-base.*

The following result appears in Tkachuk [14]. For the sake of completeness we give a proof, which is different from the one in [14].

Lemma 5.3. *Assume $K \in \mathrm{L}\Sigma(\leqslant \omega)$ is compact. Then K has a dense set of G_δ-points.*

Proof. It suffices to show that K contains at least one G_δ-point, since the class $\mathrm{L}\Sigma(\leqslant \omega)$ is stable under closed subsets. Let $\mathcal{C}$ be a compact cover of K consisting of metrizable sets and let $\mathcal{N} = \{N_n \colon n \in \omega\}$ be a countable network for $\mathcal{C}$ which consists of closed sets. Choose a sequence of open sets $\{U_n \colon n \in \omega\}$ such that $\operatorname{cl} U_{n+1} \subseteq U_n$ and either $U_n \subseteq N_n$ or $U_n \cap N_n = \emptyset$. Let $F = \bigcap_{n\in\omega} U_n$. Then F is a nonempty closed G_δ-set. Choose $C \in \mathcal{C}$ such that $F \cap C \neq \emptyset$. If $F \not\subseteq C$, then there is an $N \in \mathcal{N}$ such that $C \subseteq N$ and $F \not\subseteq N$, which is impossible by the construction of F. Thus, $F \subseteq C$. It follows that F is a closed G_δ-set contained in a metrizable subspace C of K. Hence, every point of F is G_δ in F, and therefore also G_δ in K. □

The rest of this section is devoted to the proof that the class $\mathrm{KL}\Sigma(< \omega)$ is absolute.

Assume $\varphi \colon X \to Y$ is a usc compact-valued function with nonempty images of points, and let $\mathbb{K} = \mathrm{Closed}(X)$, $\mathbb{L} = \mathrm{Closed}(Y)$. Define $h \colon \mathbb{L} \to \mathbb{K}$ by setting

$$h(a) = \{x \in X \colon \varphi(x) \cap a \neq \emptyset\}.$$

Note that h is well defined by the upper semicontinuity of φ. We will call h the *lattice map associated* to φ. Assume further that X is compact and $Y = \bigcup_{x\in X} \varphi(x)$. Observe that h has the following properties:

(1) $h(a_1 \cup a_2) = h(a_1) \cup h(a_2)$.
(2) $h(\emptyset) = \emptyset$ and $h(Y) = X$.
(3) $h(a) \neq \emptyset$ whenever $a \neq \emptyset$.
(4) If $h(a) \cap b = \emptyset$ then there exists c such that $a \cap c = \emptyset$, $b \subseteq h(c)$ and $b \cap h(a') = \emptyset$ whenever $c \cap a' = \emptyset$.

Properties (1)–(3) are obvious. To see (4), take $c = \bigcup_{x\in b} \varphi(x)$ and note that by usc, c is a compact set, and therefore is in $\mathbb{L}$.

It turns out that it is possible to reconstruct φ from a map $h \colon \mathbb{L} \to \mathbb{K}$ satisfying (1)–(4). More precisely:

Lemma 5.4. *Assume $\mathbb{L}$, $\mathbb{K}$ are normal lattices and $h\colon \mathbb{L} \to \mathbb{K}$ is a map satisfying conditions (1)–(4) above. Then there exists a compact-valued usc map $\varphi\colon \operatorname{Ult}(\mathbb{K}) \to \operatorname{Ult}(\mathbb{L})$ such that*

$$(*) \qquad h(a)^+ = \{p \in \operatorname{Ult}(\mathbb{K})\colon \varphi(p) \cap a^+ \neq \emptyset\}.$$

Proof. Fix $p \in \operatorname{Ult}(\mathbb{K})$ and define

$$I(p) = \{a \in \mathbb{L}\colon h(a) \notin p\}.$$

Observe that $I(p) = h^{-1}[p]$ is an ideal by (1), (2). Denote by "$a \ll b$" the relation "$a^+ \subseteq \operatorname{int} b^+$", which by normality can be defined algebraically as "there exists c such that $a \cdot c = 0$ and $c + b = 1$". We claim that $I(p)$ is $\ll$-directed, i.e. for every $a \in I(p)$ there is $a' \in I(p)$ with $a \ll a'$.

Fix $a \in I(p)$. Find $b \in p$ with $h(a) \cdot b = 0$. Using (4) find c such that $a \cdot c = 0$, $b \leqslant h(c)$ and $b \cdot h(a') = 0$ whenever $c \cdot a' = 0$. By normality, there are a', c' such that $a' + c' = 1$ and $a \cdot c' = 0 = a' \cdot c$. It follows that $a \ll a'$ and $h(a') \notin p$ because $h(a') \cdot b = 0$. Thus $a' \in I(p)$.

The property that $I(p)$ is $\ll$-directed ensures that the set

$$u(p) = \bigcup \{a^+ \colon h(a) \notin p\}$$

is open. Define

$$\varphi(p) = \operatorname{Ult}(\mathbb{L}) \setminus u(p).$$

We need to check that $(*)$ holds and that φ is usc.

Fix $p \in h(a)^+$ and suppose $\varphi(p) \cap a^+ = \emptyset$. Then $a^+ \subseteq u(p)$ which, by the fact that $I(p)$ is a $\ll$-directed ideal, implies that $a \in I(p)$, a contradiction. Conversely, if $\varphi(p) \cap a^+ \neq \emptyset$, then certainly $a \notin I(p)$, which means that $p \in h(a)^+$. This shows $(*)$.

To see that φ is usc, fix an open set $u \subseteq \operatorname{Ult}(\mathbb{L})$ and fix p such that $\varphi(p) \subseteq u$. Using the fact that $\{a^+ \colon a \in \mathbb{L}\}$ is a basic lattice for $\operatorname{Ult}(\mathbb{L})$, find $c \in \mathbb{L}$ such that $u \cup c^+ = \operatorname{Ult}(\mathbb{L})$ and $\varphi(p) \cap c^+ = \emptyset$. Then

$$p \in \{x \in \operatorname{Ult}(\mathbb{K})\colon \varphi(x) \cap c^+ = \emptyset\} \subseteq \{x \in \operatorname{Ult}(\mathbb{K})\colon \varphi(x) \subseteq u\},$$

and the set in the middle is open by $(*)$. It follows that φ is usc.

This completes the proof. $\square$

Lemma 5.5. *Assume $\mathbb{L}$, $\mathbb{K}$ are basic lattices for compact spaces Y and X respectively and $h\colon \mathbb{L} \to \mathbb{K}$ is a map associated to a compact-valued function $\varphi\colon X \to Y$. Define*

$$T_h = \{s \in [\mathbb{L}]^{<\omega}\colon (\forall\, a_1, a_2 \in s)\ a_1 \cap a_2 = \emptyset \text{ and } \bigcap_{a \in s} h(a) \neq \emptyset\}.$$

Endow T_h with a strict partial order $<$ defined by $s < t$ iff

(i) *every $a \in s$ is below some $a' \in t$,*
(ii) *for every $b \in t$ there is $a \in s$ with $a \subseteq b$ and*
(iii) *for some $b \in t$ there are two distinct (and therefore disjoint) sets $a_1, a_2 \in s$ such that $a_1 \cup a_2 \subseteq b$.*

Then φ is finite-valued if and only if $\langle T_h, < \rangle$ is well-founded.

Proof. Assume that $s_1 > s_2 > \dots$ is an infinite decreasing sequence in $\langle T_h, < \rangle$. Observe that $\bigcap_{a \in s_n} h(a) \supseteq \bigcap_{a \in s_{n+1}} h(a)$ for every $n \in \omega$ and therefore by the compactness of X, there is $x \in X$ such that $x \in \bigcap_{a \in s_n} h(a)$ for every $n \in \omega$. This means that $\varphi(x) \cap a \neq \emptyset$ whenever $a \in s_n$ and $n \in \omega$.

Let $F = \varphi(x) \cap \bigcap_{n \in \omega} \bigcup s_n$. By compactness and by the definition of $<$, F is an infinite closed subset of Y, which shows that $\varphi(x)$ is infinite.

Assume now that $\varphi(x)$ is an infinite set for some $x \in X$. Construct inductively a sequence $s_1 > s_2 > \dots$ in T_h such that $a \cap \varphi(x) \neq \emptyset$ for every $a \in s_n$ and $\varphi(x) \subseteq \bigcup s_n$. Since $\varphi(x)$ is infinite, $a \cap \varphi(x)$ is infinite for some $a \in s_n$, and therefore it is always possible to find $s_{n+1} < s_n$ satisfying the above condition. This shows that $\langle T_h, < \rangle$ is not well-founded. □

Theorem 5.6. *The class* $\mathrm{KL\Sigma}(< \omega)$ *is absolute. That is, if M is a transitive model of (a large enough part of) ZFC such that $X \in M$ and $M \models X \in \mathrm{KL\Sigma}(< \omega)$ then, setting $\mathbb{L} = \mathrm{Closed}(X)$ (defined in M), for every ZFC model N which extends M, we have that*

$$N \models \ \mathrm{Ult}(\mathbb{L}) \in \mathrm{KL\Sigma}(< \omega).$$

Proof. We work in M: Let T be a compact metric space and let $\varphi\colon T \to X$ be a finite-valued usc function such that $X = \bigcup_{t \in T} \varphi(t)$. Let $\mathbb{L} = \mathrm{Closed}(X)$ and $\mathbb{K} = \mathrm{Closed}(T)$. Let $h\colon \mathbb{L} \to \mathbb{K}$ be the associated map. Then the poset $\langle T_h, < \rangle$ is well founded by Lemma 5.5.

In N, h is a map of normal lattices satisfying conditions (1)–(4) (these conditions are clearly absolute), and $\langle T_h, < \rangle$ is the same object as in M. The property of being well-founded is absolute. Thus, by Lemmas 5.4 and 5.5, we deduce that $\mathrm{Ult}^N(\mathbb{L})$ is an image of a finite-valued usc function defined on $\mathrm{Ult}^N(\mathbb{K})$. It remains to observe that $\mathrm{Ult}(\mathbb{K})$ is metrizable. The metrizability of $\mathrm{Ult}(\mathbb{K})$ is equivalent to the fact that $\mathbb{K}$ is separated as a lattice by a countable family, that is, there is a countable $N \subseteq \mathbb{K}$ such that for every $a, b \in \mathbb{K}$ with $a \cdot b = 0$ there are $a', b' \in N$ such that $a \leqslant a'$, $b \leqslant b'$ and $a' \cdot b' = 0$. The last property is true in M and remains true in any ZFC model containing M.

□

Corollary 5.7. *Every space in* $\mathrm{KL\Sigma}(< \omega)$ *contains a dense metrizable subspace.*

Proof. By the above theorem combined with Lemma 5.3, Propositions 5.1, 5.2 and Corollary 2.12. □

Let us mention that, at least consistently, the class $\mathrm{KL\Sigma}(\leqslant \omega)$ is not absolute. In fact, the example of Todorčević from [18] (mentioned in the beginning of this section) constructed under $\neg\,\mathrm{MA}_{\omega_1}$, which is a metrizably fibered compact space, must have uncountable tightness after forcing with its regular-open algebra — otherwise it would contain a dense metrizable subspace.

6. Results under MA_{ω_1}

We prove that MA_{ω_1} implies that each compact spaces of scattered height 3 and of size ω_1 is in $\mathrm{L\Sigma}(\leqslant 3)$. We first state and prove a general lemma about strongly almost disjoint families

on ω_1. By a strongly almost disjoint family of sets we mean any collection of infinite sets with pairwise intersections finite. The family may contain both countable and uncountable sets.

Lemma 6.1. *Suppose that $\{A_\alpha : \alpha < \omega_1\}$ is a strongly almost disjoint family of subsets of ω_1. Suppose also that $\{p_\alpha : \alpha < \omega_1\}$ is a sequence of pairwise disjoint finite subsets of ω_1. Then there are $\alpha < \beta$ such that $p_\alpha \cap A_\beta = \emptyset$ and $p_\beta \cap A_\alpha = \emptyset$.*

Proof. By passing to a subsequence, we may assume that there is $n \in \omega$ such that $|p_\alpha| = n$ for all α. Let M be a countable elementary submodel containing everything relevant and let $\gamma = M \cap \omega_1$.

Claim 6.2. *There are $\{\alpha_i : i < n+1\} \subseteq \gamma$ and a $\beta > \gamma$ such that*

$$\Big(\bigcup_{i<n+1} p_{\alpha_i}\Big) \cap A_\beta = \emptyset$$

Proof. If not, then for each $\beta > \gamma$ there are at most n α's below γ such that $p_\alpha \cap A_\beta = \emptyset$. Thus there is a $\alpha_\beta < \gamma$ such that $p_\alpha \cap A_\beta \neq \emptyset$ for each $\alpha \in M \setminus \alpha_\beta$. Choose $\{\beta_i : i < n+1\} \subseteq M \setminus \gamma$. Choose an index $\alpha \in M$ above $\{\alpha_{\beta_i} : i < n+1\}$ such that all ordinals in p_α lie above the following finite set:

$$\bigcup_{0 \leqslant i < j < n+1} (A_{\beta_i} \cap A_{\beta_j}) \cap M$$

Then for $i < n+1$ we have that $p_\alpha \cap A_{\beta_i} \neq \emptyset$ and the sets $p_\alpha \cap A_{\beta_i}$ are disjoint. This contradicts that $|p_\alpha| = n$. $\square$

To complete the proof of the main lemma, note that since $A_{\alpha_i} \cap A_{\alpha_j} \subseteq M$ for each $i \neq j$, by the pigeon-hole argument just presented in the proof of Claim 6.2, we may conclude that $p_\beta \cap A_{\alpha_i} = \emptyset$ for some $i < n+1$ (here we use that $p_\beta \cap M = \emptyset$ for $\beta > \gamma$, and this can be easily arranged by going to a subsequence). $\square$

Theorem 6.3. MA_{ω_1} *implies that all compact spaces of cardinality ω_1 and scattered of height 3 are in* $\mathrm{L}\Sigma(\leqslant 3)$.

Proof. Let X be a compact space of scattered height 3 and of cardinality ω_1. Without loss of generality, X is of the form $I \cup D \cup \{\infty\}$ where $I = \omega_1$ is the set of isolated points and $D = \{d_\alpha : \alpha < \omega_1\}$ the set of isolated points in $X \setminus \omega_1$ (the cases where either I or D is countable do not require any extra set-theoretic assumptions and the space is in $\mathrm{L}\Sigma(\leqslant 2)$).
For each $\alpha < \omega_1$ let U_α be a clopen neighborhood of the point d_α such that $U_\alpha \setminus I = \{d_\alpha\}$. Let $a_\alpha = U_\alpha \cap \omega_1$. Then $\{a_\alpha : \alpha < \omega_1\}$ is a strongly almost disjoint family of subsets of ω_1, i.e., $a_\alpha \cap a_\beta$ is finite for each $\alpha \neq \beta$. Some, but not necessarily all, of the sets a_α may be uncountable. We may assume that

(a) For each finite $F \subseteq \omega_1$, $\omega_1 \setminus \bigcup_{\alpha \in F} a_\alpha$ is uncountable.

If not, it easy to see that X is the sum of a countable subspace and a compact subspace with finitely many non-isolated points, and therefore (without any extra set-theoretic assumptions) X is in $\mathrm{L}\Sigma(\leqslant 2)$. If we choose the neighborhoods U_α with some care, we may assure that

(b) for each finite $x \subseteq \omega_1$ and each $\alpha < \omega_1$, x is covered by $\bigcup_{\beta > \alpha} a_\beta$.

Therefore, ∞ has a local base consisting of sets of the form

$$\{\infty\} \cup (D \setminus F) \cup (\omega_1 \setminus \bigcup_{\alpha \in F} a_\alpha)$$

We may also make sure that

(c) for each finite $x \subseteq \omega_1$ there are uncountably many α such that $a_\alpha \cap x = \emptyset$.

We now define a poset $\mathbb{P}$ consisting of all pairs (p, F) where both $p, F \in [\omega_1]^{<\omega}$ with the property that $p \cap \bigcup_{\alpha \in F} a_\alpha = \emptyset$. We define $(p, F) < (q, G)$ if $q \subseteq p$ and $G \subseteq F$.

Claim 6.4. *$\mathbb{P$ has the ccc.*

Proof. Let $\{(p_\alpha, F_\alpha) \colon \alpha < \omega_1\}$ be an uncountable subset of $\mathbb{P}$. By a standard Δ-system argument, we may assume that the sets p_α's are pairwise disjoint and likewise for the F_α's. Let $A_\alpha = \bigcup_{\beta \in F_\alpha} a_\beta$. By Lemma 6.1, we may find $\alpha < \beta$ such that $p_\alpha \cap A_\beta = \emptyset = p_\beta \cap A_\alpha$. It follows that $(p_\alpha \cup p_\beta, F_\alpha \cup F_\beta) \in \mathbb{P}$ is a common extension. $\square$

Let $\mathbb{P}_\omega$ denote the finite support product of countably many copies of $\mathbb{P}$. By MA_{ω_1}, $\mathbb{P}_\omega$ has the ccc, and we may fix a filter G in $\mathbb{P}_\omega$ generic for the following dense sets:

$$E_{\alpha,G} = \{\, r \in \mathbb{P}_\omega : r(n) = (p, F) \text{ for some } n \in \omega,\ p \ni \alpha \text{ and } F \supset G \,\}$$

for $\alpha \in \omega_1$ and $G \subseteq \omega_1$ finite such that $\alpha \notin \bigcup_{\beta \in G} a_\beta$.

One easily defines from the generic two countable families of subsets of ω_1, $\{I_n \colon n \in \omega\}$ and $\{D_n \colon n \in \omega\}$ such that

(d) $\omega_1 = \bigcup_n I_n = \bigcup_n D_n$

(e) For each n, $I_n \cap a_\alpha = \emptyset$ for each $\alpha \in D_n$

(f) For each $\alpha \in \omega_1$ and each finite $F \subseteq \omega_1$ such that $\alpha \notin \bigcup_{\beta \in F} a_\beta$, there is an n such that $\alpha \in I_n$ and $F \subseteq D_n$.

We need to define a similar family of sets before we give the cover and network witnessing that $X \in \mathrm{L}\Sigma(\leqslant 3)$. To do this we let $a'_\alpha = a_\alpha \cap \bigcup_{\beta < \alpha} a_\beta$ for each α. Thus,

(g) $\{a_\alpha \setminus a'_\alpha \colon \alpha \in \omega_1\}$ is a disjoint family and $\bigcup\{a_\alpha \setminus a'_\alpha \colon \alpha \in \omega_1\} = \omega_1$.

Now we define another poset $\mathbb{Q}$ consisting of all pairs (p, F) where $p, F \in [\omega_1]^{<\omega}$ with the property that $|p \cap a'_\alpha| = 1$ and $p \cap a_\alpha = p \cap a'_\alpha$ for each $\alpha \in F$. I.e., for each $\alpha \in F$, p intersects a_α at exactly one point, and that point is in a'_α.

We take the same ordering given by $\supseteq$ on both coordinates.

Claim 6.5. *$\mathbb{Q}$ has the ccc.*

Proof. Given an uncountable subset $\{(q_\alpha, G_\alpha) \colon \alpha \in \omega_1\}$, we may assume that the q_α's form a Δ-system with root r. Let $p_\alpha = q_\alpha \setminus r$. We may also assume that the G_α's form a Δ-system with root R. Let $F_\alpha = G_\alpha \setminus R$. By going to a subsequence we may assume that $p_\alpha \cap \bigcup\{a'_\xi \colon \xi \in R\} = \emptyset$ for each α. Thus, for each $\xi \in R$, $q_\alpha \cap a'_\xi = r \cap a'_\xi$. Thus, if we let $A_\alpha = \bigcup_{\xi \in F_\alpha} a_\xi$ it follows that (q_α, G_α) is compatible with some (q_β, G_β) if and only if $p_\alpha \cap A_\beta = \emptyset = p_\beta \cap A_\alpha$. The existence of such a pair is given by Lemma 6.1. $\square$

Again we take the finite support product of countably many copies of $\mathbb{Q}$ and denote it by $\mathbb{Q}_\omega$. Taking dense sets defined similarly as above, MA_{ω_1} gives us two countable families $\{J_n : n \in \omega\}$ and $\{E_n : n \in \omega\}$ such that

(h) $|J_n \cap a_\alpha| = |J_n \cap a'_\alpha| = 1$ for each $n \in \omega$ and for each $\alpha \in E_n$.
(i) For each $\beta \in \omega_1$ and for each finite $F \subseteq \omega_1$ such that $\beta \in a'_\alpha$ for each $\alpha \in F$, there is an n such that $\beta \in J_n$ and $F \subseteq E_n$.

We are now ready to describe a cover C of the space X consisting of 3-element sets and a countable family of subsets that is a network for the cover.

For each $\alpha \in \omega_1$ and each $\beta \in a_\alpha \setminus a'_\alpha$ put $\{\beta, d_\alpha, \infty\} \in C$. Also put $\{d_\alpha, \infty\} \in C$ in the case that $a_\alpha = a'_\alpha$. Clearly, by clause (g), this collection of sets covers X. Now define a countable family of sets as follows: Let N_D be a countable family of subsets of D that separates points from finite sets. Then let $\mathcal{N}$ consist of all sets of the following forms:

(1) $\{\infty\} \cup N \cup (I_n \cap J_m)$ for $N \in N_D$ and $n, m \in \omega$.
(2) $\{\infty\} \cup N \cup I_n$ for $N \in N_D$ and $n \in \omega$,
(3) $\{\infty\} \cup N \cup J_n$ for $N \in N_D$ and $n \in \omega$, and
(4) $\{\infty\} \cup N$, for $N \in N_D$.

To see that this countable set is a network at C, fix $c = \{\beta, d_\alpha, \infty\} \in C$ and fix an open set $U \supseteq c$ (if c is of the form $\{d_\alpha, \infty\}$ the proof is easier). By clause (b), there is a finite $H \subseteq \omega_1$ with $\alpha \notin H$ such that

$$V_H = \{\infty\} \cup (D \setminus H) \cup \big(\omega_1 \setminus \bigcup_{\xi \in H} a_\xi\big) \subseteq U.$$

Thus, $c \subseteq V_H \cup \{\beta\} \subseteq U$. Let $F = \{\xi \in H : \beta \notin a_\xi\}$ and let $G = \{\xi \in H : \beta \in a_\xi\}$. Assume that F and G are both not empty (in the case that one is empty, the proof is similar). Note that since $\beta \in a_\alpha \setminus a'_\alpha$, it follows that $\beta \in a'_\xi$ for each $\xi \in G$. Fix n such that $\beta \in I_n$ and $F \subseteq D_n$, and fix m such that $\beta \in J_m$ and $G \subseteq E_m$. Then it is straightforward to verify that $\beta \in I_n \cap J_m \subseteq V_H \cup \{\beta\}$. Also fix $N \in N_D$ such that $\alpha \in N$ and $N \cap H = \emptyset$. So, $c \subseteq \{\infty\} \cup N \cup (I_n \cap J_m) \subseteq U$ as required. This completes the proof of the theorem. $\square$

We now show that some assumption is needed for Theorem 6.3. Consider the following $\clubsuit$-like principle:

($*$) There is an almost disjoint family $\{a_\alpha : \alpha \in \omega_1\}$ of countable subsets of ω_1 such that for each countable family $X \subseteq [\omega_1]^{\omega_1}$ there are uncountably many α such that $a_\alpha \cap x$ is infinite for each $x \in X$.

It is straightforward to obtain such a family from, for example, $\diamondsuit$. ($*$) gives a strong counterexample showing that some assumption is needed in Theorem 6.3:

Example 6.6. Assuming ($*$) there is a compact space X of cardinality ω_1 and of scattered height 3 such that X is not in the class $\mathrm{L}\Sigma(\leqslant \omega)$.

Proof. Let X be the one-point compactification of the Ψ-like space based on the almost disjoint family $\{a_\alpha : \alpha \in \omega_1\}$ witnessing ($*$), then X is not in $\mathrm{L}\Sigma(\leqslant \omega)$. To see this suppose otherwise, and let C be a cover by second countable (hence countable) compact subsets and

$\mathcal{N}$ a countable network with respect to C. Choose $\beta \in \omega_1$ so that if $N \in \mathcal{N}$ and if $N \cap \omega_1$ is countable then $N \cap \omega_1 \subseteq \beta$. Enumerate the set $\{N \cap (\omega_1 \setminus \beta) \colon N \in \mathcal{N}\} \setminus \{\emptyset\}$ as $\{N_k \colon k \in \omega\}$. Choose now a $c \in C$ such that $c \cap (\omega_1 \setminus \beta)$ is not empty. Since c is countable, we may also choose α such that $a_\alpha \notin c$ and $a_\alpha \cap N_k$ is infinite for each $k \in \omega$. But $a_\alpha \cap c$ is finite, so $U = X \setminus (\{a_\alpha\} \cup (a_\alpha \setminus c))$ is an open set containing c. However, by choice of a_α, there is no element of $\mathcal{N}$ containing c and contained in U. This contradicts that $\mathcal{N}$ is a network at C. $\square$

Theorem 6.7. *Assuming* MA_{ω_1}, *Aronszajn trees are in the class* $\mathrm{L}\Sigma(\leqslant \omega)$.

Proof. For the topology on T, an Aronszajn tree, we define $[s] = \{t \in T \colon s \leqslant t\}$ and declare $[s]$ clopen for each s of successor height in T. For each $s \in T$ let $\lceil s \rceil = \{t \in T \colon t \leqslant s\}$, the *downward closure of s*. The downward closure of each element of T is compact second countable, so if $\mathcal{C} = \{\lceil s \rceil \colon s \in T\}$ then $\mathcal{C}$ is a cover by second countable compact sets. For $X \subseteq T$ we let $\lceil X \rceil = \bigcup \{\lceil t \rceil \colon t \in X\}$.

We will apply MA_{ω_1} to find a family $\{(D_n, F_n) \colon n \in \omega\}$ such that

(1) $\lceil D_n \rceil \cap F_n = \emptyset$ for each $n \in \omega$, and
(2) for each $t \in T$ and each $F \in [T]^{<\omega}$ such that $\lceil t \rceil \cap F = \emptyset$, there is n such that $t \in D_n$ and $F \subseteq F_n$.

Indeed, if such a family exists, it is straightforward to verify that $\{\lceil D_n \rceil \colon n \in \omega\}$ is a network for the cover $\mathcal{C}$.

To obtain the family, we first define a poset $\mathbb{P}$ as follows. Let

$$\mathbb{P} = \{(x, F) \colon x, F \in [T]^{<\omega} \text{ such that } \lceil x \rceil \cap F = \emptyset\}.$$

We order $\mathbb{P}$ in the natural way: $(x, F) < (y, G)$ if $y \subseteq x$ and $F \subseteq G$.

Claim 6.8. $\mathbb{P}$ *is ccc*

Proof. Let $\{(x_\alpha, F_\alpha) \colon \alpha \in \omega_1\} \subseteq \mathbb{P}$. Without loss of generality we may assume that

(a) for $\alpha < \beta$, all elements of $x_\alpha \cup F_\alpha$ lie in levels lower than any element of $x_\beta \cup F_\beta$,
(b) there is $n, m \in \omega$ such that $|x_\alpha| = n$ and $|F_\alpha| = m$ for all $\alpha \in \omega_1$.

We prove by induction on n and m that there is an uncountable centered subset. The first nontrivial case is where $n = m = 1$. In this case let $x_\alpha = \{s_\alpha\}$ and $F_\alpha = \{t_\alpha\}$. Since we are assuming MA_{ω_1} we have that T is special, so we may assume that

(c) $\{t_\alpha \colon \alpha \in \omega_1\}$ is an antichain in T.

If $\{\beta \colon t_\alpha < s_\beta\}$ is countable for each α, then it is straightforward to recursively construct an uncountable pairwise compatible subset. Otherwise, there is α_0 such that $A = \{\beta \colon t_{\alpha_0} < s_\beta\}$ is uncountable. However, in this case (a) and (c) imply that $\{(s_\beta, t_\beta) \colon \beta \in A\}$ is pairwise compatible (in fact centered).

To accomplish the induction step assume that for any family $\{(y_\alpha, G_\alpha) \colon \alpha \in \omega_1\}$ as above with $|y_\alpha| = n - 1$ has an uncountable centered family and that for any family $\{(y_\alpha, G_\alpha) \colon \alpha \in \omega_1\}$ with $|y_\alpha| = n$ and $|G_\alpha| < m$ has an uncountable centered family.

Consider first the case that $m > 1$: Fix $t_\alpha \in F_\alpha$ and let $G_\alpha = F_\alpha \setminus \{t_\alpha\}$. By the inductive assumption for $m-1$, there is an uncountable A such that $\{(x_\alpha, G_\alpha)\colon \alpha \in A\}$ is centered. Also by the induction assumption for $1 < m$ there is an uncountable $B \subseteq A$ such that $\{(x_\alpha, \{t_\alpha\})\colon \alpha \in B\}$ is centered. By the definition of the ordering it follows that $\{(x_\alpha, F_\alpha)\colon \alpha \in B\}$ is centered.

To prove the case where $m = 1$, we may assume that $n > 1$. Fix $s_\alpha \in x_\alpha$ and let $y_\alpha = x_\alpha \setminus \{s_\alpha\}$. By the same procedure as above we may obtain an uncountable B so that both families $\{(y_\alpha, F_\alpha)\colon \alpha \in B\}$ and $\{(\{s_\alpha\}, F_\alpha)\colon \alpha \in B\}$ are centered. By the definition of the ordering it follows that $\{(x_\alpha, F_\alpha)\colon \alpha \in B\}$ is centered. $\square$

To finish the proof of the theorem, take the finite support product of countably many copies of $\mathbb{P}$. Take G generic for the family of dense sets: $D_{t,G} = \{p\colon (\exists\, n)\; p(n) = (x, F),\ t \in x, G \subseteq F\}$ where (t, G) range over all pairs $t \in T$ and $G \in [T]^{<\omega}$ such that $\lceil t \rceil \cap G = \emptyset$. Letting

$$D_n = \bigcup \{x : \exists p \in G \exists F (p(n) = (x, F)\}$$

and

$$F_n = \bigcup \{F : \exists p \in G \exists x (p(n) = (x, F)\}$$

it is easy to verify that $\{(D_n, F_n) : n \in \omega\}$ satisfy 1. and 2. as required.

$\square$

7. Open questions

Question 7.1. Does there exist in ZFC a space in $\mathrm{KL\Sigma}(\leqslant \omega)$ without a dense metrizable subspace?

Question 7.2. Is the class of compact spaces in $\mathrm{L\Sigma}(< \omega)$ absolute?

Question 7.3. Does there exist a (σ-compact) space X such that $\mathrm{t}(X) > \omega$ and $X \in \mathrm{L\Sigma}(n)$ for some $n \in \omega$?

Question 7.4. Assume that $X^\omega \in \mathrm{L\Sigma}(< \omega)$. Is it true in ZFC that $nw(X) \leqslant \omega$?

Question 7.5. Assume that $X \in \mathrm{L\Sigma}(\leqslant \omega)$ and $p\colon X \to Y$ is a finite-valued usc mapping, $Y = p(X)$. Must Y belong to $\mathrm{L\Sigma}(\leqslant \omega)$?

Question 7.6. Are all Rosenthal compacta in $\mathrm{L\Sigma}(\leqslant \omega)$?

Question 7.7. Assume MA_{ω_1}. Is it true that every scattered compact space of cardinality ω_1 and height n, $n \in \omega$, belongs to $\mathrm{L\Sigma}(\leq n+1)$?

Question 7.8. Suppose that $X \in \mathrm{L\Sigma}(\leqslant \omega)$ and Y is a space such that $C_p(X)$ is homeomorphic to $C_p(Y)$. Must Y belong to $\mathrm{L\Sigma}(\leqslant \omega)$?

Remark. If $X \in \mathrm{L\Sigma}(< \omega)$ and $C_p(X)$ is homeomorphic to $C_p(Y)$, then $Y \in \mathrm{L\Sigma}(< \omega)$; this follows from the invariance of $\mathrm{L\Sigma}(< \omega)$ with respect to finite-valued usc images and the main theorem in [11]. By a similar argument, a positive answer to Question 7.5 implies “yes” for Question 7.8.

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